Polarization in Strategic Networks

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Abstract

A model of social learning and strategic network formation is developed with distance-based utility and *cognitive dissonance*. For intermediate costs, stable networks exhibit realistic properties and belief polarization increases with small increases in available information.

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1 Introduction

Individuals form beliefs in many ways, including listening to the beliefs of their friends and acquaintances. A natural consequence of this is that networks are homophilic in beliefs (MacPherson, Smith-Lovin, and Cook, 2001), and can often lead to outright agreement (Golub and Jackson, 2010). On the other hand, individuals have a tendency to avoid seriously listening to opinions that diverge from their own due to psychological discomfort. Such a phenomenon is known as *cognitive dissonance* (Festinger, 1962), and, while also a force for homophily within groups, is a force for disagreement across groups.

This note develops a model with social learning similar to DeGroot (1974), and endogenous network formation similar to Jackson and Wolinsky (1996). Cognitive dissonance is present, and each agent has a bias. The main results are for intermediate costs. First, there is an equilibrium network with realistic properties, such as high clustering and low diameter. Furthermore, this equilibrium network has two dense groups connected by a bridge, and produces heterogenous beliefs. Newman (2010) and Jackson (2008) provide surveys of the networks literature.

Second, in this equilibrium the dispersion of opinions increase in the size of the network. This second point is of particular interest: increasing the amount of information in a network increases the dispersion of opinions. Other models of social learning with endogenous network formation (Arifovic, Eaton, and Walker, 2015; Holme and Newman, 2006) do not produce this result.

The paper is structured as follows. Section 2 outlines the learning and network formation processes. Section 3 presents results. Section 4 concludes.

2 Model

2.1 Learning

Let the set of agents be $\mathbf{N}$ and the undirected network adjacency matrix be $\mathbf{A}$. Time is discrete, where $t \in \mathbb{Z}_+$. Each agent $i \in \mathbf{N}$ has a time t belief p_i^t and a bias x_i. In general, an agent's beliefs are updated according to

$$p_i^{t+1} = \alpha x_i + (1-\alpha)\frac{\sum_{j \in \mathbf{N}} a_{ij} p_j^t}{\sum_{k \in \mathbf{N}} a_{ij}} \tag{1}$$

where a_{ij} are elements of $\mathbf{A}$. The entire system of beliefs, given the network structure, is in steady-state when $p_i^t = p_i^{t+1}$ for all $i \in \mathbf{N}$ and $t \in \mathbb{Z}_+$. The solution is given by:

$$\mathbf{p} = \alpha[1 - (1-\alpha)\mathbf{G}]^{-1}\mathbf{x} \tag{2}$$

where $\mathbf{G}$ is the appropriately modified adjacency matrix. This term gives an individual's beliefs as his *alpha centrality* with parameter $(1-\alpha)$. The term $1-(1-\alpha)\mathbf{G}$ is invertible if and only if $1 - \alpha < \frac{1}{\lambda_{\max}}$, where λ_{max} is the largest eigenvalue of

$\mathbf{G}$. Because $\mathbf{G}$ is row-stochastic, its largest eigenvalue is at most 1. Therefore, a steady-state belief exists for $\alpha > 0$.

Golub and Jackson (2010) show that for $\alpha = 0$ the system converges to $p_i = p_j$ when the network structure satisfies certain conditions.[1] When $\alpha > 0$ and $x_i \neq x_j$ for some i and j then agreement will not occur.

The goal is to characterize (2) for an endogenous network $\mathbf{A}(\mathbf{p})$, which depends on $\mathbf{p}$.

2.2 Network Formation

Consider the following utility function:

$$V_i(\mathbf{A}, \mathbf{p}) = \sum_{j \neq i} \delta^{d_{ij}-1} - a_{ij} c |p_i - p_j| \tag{3}$$

where $\delta \in [0, 1)$, $c \geq 0$, and d_{ij} is the geodesic distance[2] between i and j. The utility function (3) is a modification of Jackson and Wolinsky (1996). The difference is that the costs of direct links are proportional to the differences in belief. The parameter c captures the degree to which agents experience cognitive dissonance.

Given a set of beliefs, one can define a notion of stability for this network. The concept employed here is *pairwise stability*. Let $\mathbf{A} + ij$ be the network $\mathbf{A}$ with the link ij added, and $\mathbf{A} - ij$ be the network $\mathbf{A}$ with the link ij removed.

Definition 2.1 (Pairwise Stability)

A network $\mathbf{A}(\mathbf{p})$ *is* pairwise stable *if for all* i *and* j*:*

(i) for all links ij *not in the network,* $V_i(\mathbf{A}(\mathbf{p}) + ij, \mathbf{p}) > V_i(\mathbf{A}(\mathbf{p}), \mathbf{p})$ *implies* $V_j(\mathbf{A}(\mathbf{p}) + ij, \mathbf{p}) < V_j(\mathbf{A}(\mathbf{p}), \mathbf{p})$

(ii) for all links ij *in the network,* $V_i(\mathbf{A}(\mathbf{p}) - ij, \mathbf{p}) \leq V_i(\mathbf{A}(\mathbf{p}), \mathbf{p})$ *and* $V_j(\mathbf{A}(\mathbf{p}) - ij, \mathbf{p}) \leq V_j(\mathbf{A}(\mathbf{p}), \mathbf{p})$

Notice that the definition depends on the beliefs in place. Therefore, a network is pairwise stable if, given beliefs, adding a link makes one of the nodes strictly worse off and removing links makes both nodes weakly worse off.

2.3 Solution Concept

We need a stronger definition of equilibrium than pairwise stability to characterize both belief formation and network formation. The following equilibrium will be used for the analysis.

[1] *Strongly connected* and *closed* groups converge to agreement. In the undirected case, strongly connected is equivalent to connected. Furthermore, they require $\mathbf{G}$ to be aperiodic.

[2] The geodesic distance is the shortest path between i and j, and must take values on $\mathbb{Z}_{++}$.

Definition 2.2 (Stability)

A network and belief pair $\eta^* \equiv (\mathbf{A}(\mathbf{p}^*), \mathbf{p}^*)$ *is stable if* $\mathbf{A}(\mathbf{p}^*)$ *is pairwise stable and* $\mathbf{p}^*$ *satisfies (2).*

Therefore, η^* requires having pairwise stability, and beliefs are in steady-state. This equilibrium notion is intended to analyze the long-run behaviour of the model.

3 Intermediate Cost Equilibrium

First, it is assumed that $|\mathbf{N}|$ is even, and there are two groups of agents of size $N = \frac{|\mathbf{N}|}{2}$. Each group has a common bias, which are $x_1 < x_2$. Furthermore, it is assumed that agents weight their bias and friends beliefs equally[3] so that $\alpha = \frac{1}{1+\sum_j a_{ij}}$.

The intermediate cost case is similar to (though not a special case of) the islands network described in Jackson and Rogers (2005). The main differences are that types are endogenous and heterogeneous within islands.

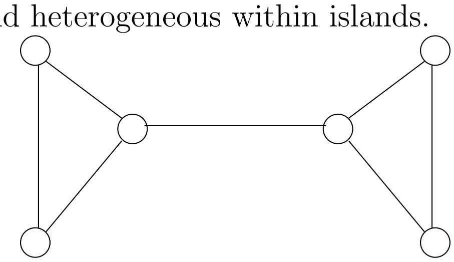

Figure 1: Islands Network with $N = 3$.

The topology has all agents with the same biases connected. However, there is one agent with bias x_1 and one agent with bias x_2 that also connect. These two agents have slightly different beliefs than the rest of their neighbourhood, and "pull" the beliefs toward the centre. Figure 1 illustrates the topology for the case of $N = 3$.

Proposition 3.1 (Islands Network)

Suppose that $(1-\delta^2)(1+\delta(N-1))\left(\frac{N+5}{N+2}\right) < c(x_2 - x_1) < \left(\frac{N+5}{N+1}\right)(1+\delta(N-1))$ *and* $c(x_2 - x_1) < (1-\delta)(N+5)$*. A stable network exists such that:*

(i) $x_i = x_j \Rightarrow a_{ij} = 1$,

(ii) $\exists$ *exactly one* i *with prior* x_1 *and one* j *with prior* x_2 *s.t.* $a_{ij} = 1$,

(iii) Those with prior x_i *and degree* $(N-1)$ *have beliefs* $p_i = \left(\frac{N+4}{N+5}\right) x_i + \left(\frac{1}{N+5}\right) x_j$,

(iv) Those with prior x_i *and degree* N *have beliefs* $p_i = \left(\frac{N+3}{N+5}\right) x_i + \left(\frac{2}{N+5}\right) x_j$.

[3] This is the dissonance-minimizing weight when dissonance is $(p_i - x_i)^2 + \sum_j a_{ij}(p_i - p_j)^2$.

Proof:

There are four "types" of agents here, distinguished by two biases and two network positions. The four equations describing the beliefs of each "type" of agent are:

$$p(x_1) = \frac{x_1}{N} + \frac{(N-2)p(x_1)}{N} + \frac{\hat{p}(x_1)}{N} \tag{4}$$

$$\hat{p}(x_1) = \frac{x_1}{N+1} + \frac{(N-1)p(x_2)}{N+1} + \frac{\hat{p}(x_2)}{N+1} \tag{5}$$

$$p(x_2) = \frac{x_2}{N} + \frac{(N-2)p(x_2)}{N} + \frac{\hat{p}(x_2)}{N} \tag{6}$$

$$\hat{p}(x_2) = \frac{x_2}{N+1} + \frac{(N-1)p(x_2)}{N+1} + \frac{\hat{p}(x_1)}{N+1} \tag{7}$$

Solving the above system results in (*iii*) and (*iv*) of the proposition. To derive (*i*) and (*ii*) from the conditions in the proposition, we must check 14 conditions applying to the relationships between each "type" of node. Because those with each bias are treated symmetrically, this lowers the number of conditions to (at most) seven. The conditions are:

$$1 > \delta \tag{8}$$

$$1 + \delta + (N-1)\delta^2 - c|p^*(x_1) - \hat{p}^*(x_1)| > \delta + \delta^2 + (N-1)\delta^3 \tag{9}$$

$$1 + (N-1)\delta - c|p^*(x_1) - \hat{p}^*(x_2)| < \delta + (N-1)\delta^2 \tag{10}$$

$$1 + (N-1)\delta - c|p^*(x_1) - p^*(x_2)| < \delta + (N-1)\delta^2 \tag{11}$$

$$1 + (N-1)\delta - c|\hat{p}^*(x_1) - \hat{p}^*(x_2)| > 0 \tag{12}$$

$$1 - c|\hat{p}^*(x_1) - p^*(x_2)| < \delta \tag{13}$$

$$1 - c|p^*(x_1) - \hat{p}^*(x_1)| > \delta \tag{14}$$

Of the seven equations, (8), (9), (11), (12), and (14) must hold for the proposed network to be pairwise stable. However, only one of equations (10) and (13) must hold (though both may hold).

Equation (8) holds by assumption. Furthermore, (10) implies (11). It can also be verified that equation (14) implies equation (9). Therefore, sufficient conditions for pairwise stability that can be checked are given by equations (10), (12), and (14). These equations are the conditions of the proposition, after algebra.

■

The network consists of one component, two complete subnetworks determined by biases, and a "bridge" between the subnetworks. The following proposition discusses properties of this equilibrium.

Proposition 3.2 (Properties)

The stable network considered in Proposition 3.1 has the following properties.

(i) The diameter of the network is 3 for $N > 1$.

(ii) The overall clustering coefficient is $\frac{N(N-2)}{2+N(N-2)}$.

(iii) An increase in N leads to an increase in var(p_i^*).

Proof:

It is clear that (i) is true, as one can get from those with belief $p^*(x_1)$ to those with $p^*(x_2)$ in three steps. Showing (ii) involves calculating the fraction of connected tripled that are closed (which is the definition of the overall clustering coefficient). The number of closed triples is the number of connected triples in the set of agents with bias x_1 plus the number of connected triples in the set of those with bias x_2. This number is $N(N-1)(N-2)$. The total number of connected triples is the number of closed triples plus the number of non-closed connected triples. This second number is all of those triples that involve the “bridge” in the network, which is $2(N-1)$. The result (ii) follows.

To show (iii), notice that the mean equilibrium belief is always $\frac{x_1+x_2}{2}$, and $\text{var}(p^*) = \left(\frac{x_1+x_2}{4(N+5)}\right)((N-1)(N+3)^2 + (N+1)^2)$. Taking the derivative with respect to N proves (iii).

■

Notice that beliefs diverge slowly as N begins to increase. However, once N reaches a certain threshold[4] then the network reaches a tipping point. The set of network topologies that become stable after this tipping point depends on the underlying parameters of the network.

4 Conclusion

This paper develops a model of strategic network formation and social learning. When the mental costs of cognitive dissonance are of intermediate size then stable networks can exhibit realistic properties and adding agents to the society will lead to an increase in polarization. Future work should generalize the results presented here.

[4] For $N \to +\infty$ the stable η^* is not an islands topology.

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