Consensus in Social Networks: Revisited

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Abstract

We analyze the convergence of opinions or beliefs in a general social network with non-Bayesian agents. We provide a new sufficient condition under which opinions converge to consensus. Our condition is significantly more permissive than that of Lorenz (2005).

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1 Introduction

Social networks are an important source of information for individuals and firms. The emergence of social media has led to an unprecedented level of information sharing among "friends," i.e. those who are connected and communicate. Given this, should one expect people to agree in the long run? We provide a new sufficient condition under which non-Bayesian agents in a given network converge to consensus.

In our model, agents update their opinions based on the prior opinions of their friends (and potentially themselves). Though we focus on opinions, the model can accommodate any variable on a convex set, where the convex hull of initial values is compact. For instance, instead of an opinion – a subjective probability – agents may update their belief about the value of an unknown parameter, adopt a cultural norm, or best-respond to previous strategies in a game (i.e., Cournot learning).

Literature on non-Bayesian learning beginning with DeGroot (1974) has agents updating their beliefs to a weighted average of their friends' beliefs. Lorenz (2005) provides a generalization of the DeGroot model by allowing the weights depend on time and prior beliefs. The level of generality allows for many types of updating behaviour, including those that exhibit optimism or pessimism (over-weighting or under-weighting), and cognitive dissonance (giving a higher weight to those with similar beliefs). He demonstrates that aperiodic and strongly connected networks reach agreement if the weight one gives to a friend's opinion is bounded away from 0 by a positive number.[1] We provide a more permissive sufficient condition than that of Lorenz (2005). Roughly speaking, Roughly speaking, our result says that consensus is achieved unless some agents rely with an increasingly "faster" rate on their friends with the minimal opinion while some others on those with the maximal opinion.

DeMarzo et al. (2003) considers a time-varying social network that has agents' weighting themselves differently over time. They show that opinions converge when agents weight other people's opinions "often enough." Our result is related to DeMarzo et al. (2003)'s, and the two are equivalent for complete networks. Furthermore, in non-complete networks our condition is more restrictive. However, our condition is applicable in a wide range of networks while DeMarzo et al. (2003)'s condition is not applicable outside of their specific model.

Mueller-Frank (2013) considers a general class of time-varying updating rules that include rules with belief-dependent weights. The main conditions for convergence to consensus are (i) updating rules must satisfy continuity and have posteriors be strictly in between the most

[1]In particular, Lorenz (2005) requires that if there exists y and τ such that $w^{\tau}(y) \geq \delta > 0$ then $w^{t}(x) \geq \delta > 0$ for all x and t.

extreme priors in one's neighborhood and (ii) the period-by-period updating functions must be of finite type. Our result does not require updating rules to be continuous or be of finite type.

This note is structured as follows. Next we introduce preliminary concepts and notation. Section 3 contains results and examples. We conclude with a discussion. Proofs are found in the Appendix.

2 Preliminaries

A finite set $A = \{1, \cdots, a\}$ of agents interact with each other. Each agent $i \in A$ listens to a fixed group of agents, which may exclude i. A function $C : A \to 2^A$ which maps each agent to some subset of A identifies the set of agents to whom a given agent listens. Specifically, i listens to $C(i)$ and we sometimes refer to $C(i)$ as i's neighborhood. Naturally, if some agent j is in some agent i's neighborhood, we say j is i's neighbor. A pair $\langle A, C\rangle$ is a network. We denote the set of agents to whom i listens in $k \geq 2$ steps by $C^k(i)$. Formally, $C^k(i)$ is defined iteratively as follows: $C^k(i) = \cup_{j\in C(i)}C^{k-1}(j)$.

We say that agent i and j communicate if there exist natural numbers k and k' such that $j \in C^k(i)$ and $i \in C^{k'}(j)$. Network $\langle A, C\rangle$ is irreducible if any two agents in A communicate. A sequence of agents $i_1, i_2, \cdots, i_k$ is a simple cycle if (i) $i_1 = i_k$, (ii) no agent other than i_1 appears more than once in the sequence while i_1 appears exactly twice and (iii) i_l listens to i_{l+1} for all $l = 1, \cdots k - 1$. The length of a simple cycle $i_1, i_2, \cdots, i_k$ is $k - 1$.

Definition 1. *A network $\langle A, C\rangle$ is aperiodic if the greatest common divisor of the lengths of its simple cycles is 1.*

Let us fix a network $\langle A, C\rangle$ which is irreducible and aperiodic. We use the following notation:

$$\theta \equiv \arg\min_k \{k \in \mathbb{Z}_+ | C_i^\kappa = A, \forall i \in A, \forall \kappa \geq k\}.$$

It is well-known that θ exists for irreducible, aperiodic networks.

An opinion/belief of the agents is an a-dimensional vector x where x_i is agent i's opinion about some parameter. Each agent's opinion x_i is in the $[0, 1]$ interval, and consequently, the set of possible opinions is $[0, 1]^a$. We use the following conventional notations: for each $i \in A$, $x_{-i} \equiv (x_j)_{j\neq i}$ and $x = (x_i, x_{-i})$.

Time is discrete and starts at period 0. At the initial period, the agents have an exogenously given opinion, and they exchange their opinions according the network structure. Afterwards they update their opinions which become the following period's initial opinions.

In the following period, the agents again exchange and update their opinions. The process repeats every period. We formalize this opinion updating process by introducing an (opinion) updating function $T : \mathbb{N} \times [0,1]^a \to [0,1]^a$ where $\mathbb{N}$ is the set of non-negative integers. Agent i's updating function is T_i and the process is a Markov chain.[2] If the opinion is x in period t then $T(t,x)$ is the opinion in period $t+1$. We will sometimes use the notation $T^{t,1}(x)$ for $T(t,x)$ and iteratively define $T^{t,k}(x)$ as $T(t+k-1, T^{t,k-1}(x))$ for all integer $k \geq 2$. In words, $T^{t,k}(x)$ is the vector of opinions in period $t+k$ when the period t vector of opinions is x.

We are interested in how the agents' opinions evolve in the long-run. In this sense, the main focus of our study is the properties of $T^{\infty}(x) \equiv \lim_{k\to\infty} T^{0,k}(x)$ when it is well-defined. We say a network reaches consensus if $T_i^{\infty}(x) = T_j^{\infty}(x)$ for all x, i and j.

As we indicated before, the network structure must affect the updating function. Specifically, we assume that (i) one's opinion is not affected by the opinions of those who are not in the agent's neighborhood, i.e., for each x and $\bar{x}_{-C(i)}$, $T_i(t,x) = T_i(t, x_{C(i)}, \bar{x}_{-C(i)})$ for all $t \geq 0$, and (ii) if agent j is i's neighbor then j's opinion affects i's in some cases, i.e., for each $j \in C(i)$, there exists x and $\bar{x}_j$, and t such that $T_i(t,x) \neq T_i(t, \bar{x}_j, x_{-j})$. We sometimes refer to $T(t,\cdot)$ as the period-t updating function.

We assume that no agent updates her opinion outside of the extremal opinions of her neighbors.

Assumption 1. *$T_i(t,x) \in [\min_{j\in C(i)} x_j, \max_{j\in C(i)} x_j]$ for all i and x.*

Unless otherwise stated, Assumption 1 holds for the rest of this paper. Next we present some examples of updating functions, each of which satisfy Assumption 1.

$$T_i(t,x) = \left(\sum_{j\in C(i)} w_{ij}^t x_j^p\right)^{\frac{1}{p}} \qquad \text{where } w_{ij}^t > 0,\ \sum_{j\in C(i)} w_{ij}^t(x) = 1 \quad (1)$$

$$T_i(t,x) = \lambda^t \left(\sum_{j\in C(i)} w_{ij} x_j\right) + (1-\lambda^t)x_i \qquad \text{where } w_{ij} > 0,\ \sum_{j\in C(i)} w_{ij} = 1, \lambda^t \in [0,1] \quad (2)$$

$$T_i(t,x) = \frac{\Pi_{j\in C(i)} x_j^{w_{ij}^t}}{\Pi_{j\in C(i)}(1-x_j)^{w_{ij}^t} + \Pi_{j\in C(i)} x_j^{w_{ij}^t}} \qquad \text{where } w_{ij}^t > 0,\ \sum_{j\in C(i)} w_{ij}^t = 1 \quad (3)$$

$$T_i(t,x) = \sum_{j\in C(i)} w_{ij}^t(x) x \qquad \text{where } w_{ij}^t(x) > 0,\ \sum_{j\in C(i)} w_{ij}^t(x) = 1 \quad (4)$$

[2] We note here that allowing updating functions to be dependent on history of opinions does not alter our main result, Theorem 1.

The updating rule in (1) is a (weighted) L_p-norm of opinions. Notice that the weights, w^t_{ij}, vary over time. When $p = 1$ this rule reduces to the one in DeGroot (1974). The updating rule in (2) is considered in DeMarzo et al. (2003). This updating function has a very specific structure: the time-varying weight is on a constant group of friends and one's own prior. This is equivalent to varying inertia in opinions.

The updating rule in (3) is considered by Molavi et. al. (2016).[3] with time-varying weights. A recent paper by **?** studies the foundations of social learning using an axiomatic approach. This updating functions is "more Bayesian" than the standard DeGroot one in the sense that it violates fewer properties of a Bayesian updating function. Equation (4) gives the formulation of the model expressed in Lorenz (2005). Notice that the weights, $w^t_{ij}(\cdot)$, vary over time and is a function of current opinions. It is easy to see that any updating function can be written in the form of (4).

Lorenz (2005) shows that if $w^t_{ij}(x) \geq \delta > 0$ for all $t \geq 0$, $i \in A$, $j \in C(i)$ and x, then the agents' opinions converge to consensus in the long run (assuming an irreducible and aperiodic network). This sufficient condition is not satisfied for (2) when $\lambda^t \to 0$ fast enough, or for (1) when $w^t_{ij} \to 0$ for some i and $j \in C(i)$. However, in these cases consensus sometimes is reached. We will introduce a general condition that subsumes Lorenz (2005)'s sufficient condition.

3 Results

To introduce our condition, we need to define the following two variables:

$$\alpha^t_i(x) = \begin{cases} 1 & \text{if } |C(i)| = 1 \text{ or if } \max_{j\in C(i)} x_j = \min_{j\in C(i)} x_j \\ \frac{T_i(t,x) - \min_{j\in C(i)} x_j}{\max_{j\in C(i)} x_j - \min_{j\in C(i)} x_j} & \text{in all other cases} \end{cases}$$

and

$$\beta^t_i(x) = \begin{cases} 1 & \text{if } |C(i)| = 1 \text{ or if } \max_{j\in C(i)} x_j = \min_{j\in C(i)} x_j \\ \frac{\max_{j\in C(i)} x_j - T_i(t,x)}{\max_{j\in C(i)} x_j - \min_{j\in C(i)} x_j} & \text{in all other cases} \end{cases} .$$

Observe here that

$$\begin{aligned} T_i(t,x) &= (1 - \alpha^t_i(x)) \min_{j\in C(i)} x_j + \alpha^t_i \max_{j\in C(i)} x_j \\ &= \beta^t_i \min_{j\in C(i)} x_j + (1 - \beta^t_i(x)) \max_{j\in C(i)} x_j . \end{aligned}$$

[3]In Malavi et al. (2016) the weights w^t_{ij} are time-independent, and $\sum_{j\in C(i)} w^t_{ij}$ need not equal 1.

If we think of $T_i(t, x)$ as the convex combination of the extremal opinions in i's neighborhood, then $\alpha_i^t(x)$ and $\beta_i^t(x)$ are the weights i places on the maximal and minimal opinions, respectively.

Let $\underline{\alpha}^t$ be the lowest weight given by any agent to the maximal opinion in her neighborhood, i.e., $\underline{\alpha}^t \equiv \inf_{i\in A \& x\in[0,1]^a} \alpha_i^t(x)$. In addition, for any integer $k \geq 1$, let $\underline{\alpha}^{t,k} \equiv \prod_{\tau=t}^{t+k-1} \underline{\alpha}^\tau$. Similarly, we define $\underline{\beta}^t$ and $\underline{\beta}^{t,k}$. Observe here that $\underline{\alpha}^{t,k} + \underline{\beta}^{t,k} \leq 1$ for all integers $t \geq 0$ and $k \geq 1$ in irreducible networks because $\alpha_i^t(x) = 1 - \beta_i^t(x)$ for all i and x with $\max_{j\in C(i)} x_j \neq \min_{j\in C(i)} x_j$.

In the lemma below, we consider how the extremal opinions behave.

Lemma 1. *Let $\langle A, C\rangle$ be an irreducible, aperiodic network. Then for all x and $t \geq 0$,*

$$\max_{j\in A} T_j^{t,\theta}(x) - \min_{j\in A} T_j^{t,\theta}(x) \leq (1 - \underline{\alpha}^{t,\theta} - \underline{\beta}^{t,\theta}) \left(\max_{j\in A} x_j - \min_{j\in A} x_j \right).$$

If the network is complete, i.e., if $j \in C(i)$ for all i and j, then the definitions of $\underline{\alpha}^t$ and $\underline{\beta}^t$ give the lemma above with $\theta = 1$. In non-complete networks, the intuition behind the lemma is as follows: because the network is irreducible and aperiodic, all the agents communicate with one another after θ periods. This means that both maximal and minimal (initial) opinions affect each agent's opinion in θ periods. The lowest weight one assigns to the maximal opinion in her neighborhood in period τ is $\underline{\alpha}^\tau$. Thus, each agent must assign at least the weight of $\underline{\alpha}^{t,\theta}$ to the period-t maximal opinion in the whole network after θ periods. Thus, $T_i^{t,\theta}(x) \geq (1 - \underline{\alpha}^{t,\theta}) \min_{i\in A} x_j + \underline{\alpha}^{t,\theta} \max_{i\in A} x_j$ for all i. A similar logic yields that $T_i^{t,\theta}(x) \leq \underline{\beta}^{t,\theta} \min_{j\in A} x_j + (1 - \underline{\beta}^{t,\theta}) \max_{j\in A} x_j$. By rearranging terms, we obtain that between periods t and $t + \theta$, the distance between extremal opinions shrinks at least by $\underline{\alpha}^{t,\theta} + \underline{\beta}^{t,\theta}$ fraction.

Theorem 1. *Let $\langle A, C\rangle$ be an irreducible, aperiodic network. Then consensus is reached if there exists a sequence $\{t_k\}$ such that (i) $t_{k+1} - t_k \geq \theta$ for all k and (ii)*

$$\lim_{\tau\to\infty} \sum_{k=1}^{\tau} (\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}) = \infty$$

To prove this theorem, note that the extremal opinions in the network cannot move further apart over time because (by Assumption 1) no agent's updated opinion falls outside of the interval formed by the extremal opinions in the agent's neighborhood. The lemma preceding the theorem means that after τ blocks of θ periods (where block k starts at period t_k), the extremal opinions will be at most $\prod_{k=1}^{\tau}(1 - \underline{\alpha}^{t_k,\theta} - \underline{\beta}^{t_k,\theta})$ fraction of the distance between extremal opinions in the initial period. We complete the proof by showing that this maximal fraction goes to 0 as the number of blocks increases as long as the sum of $(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta})$ over k converges to infinity.

Our sufficient condition means that unless some agents rely on the minimal opinion while others on the maximal opinion at an increasingly "faster rate," consensus is reached in irreducible, aperiodic networks. It is easy to see that our condition is significantly more general than that of Lorenz (2005). He considers updating functions in the form of (4) and shows that consensus is reached if $w_{ij}^t(x) \geq \delta > 0$ for all t, i, $j \in C(i)$ and x. Obviously, our condition is only in terms of weights assigned to the extremal opinions. In fact, as long as one of these is bounded below or is converging to 0 slowly then our condition is satisfied. Consequently, our condition subsumes Lorenz's.

It is also easy to see that consensus occurs in the long term if $\underline{\alpha}^{t,\theta} + \underline{\beta}^{t,\theta} = 1$ for some t. In non-complete networks, this condition requires that either everyone updates her opinion to the maximal one in each period between t and $t+\theta$ or everyone to the minimal one. In complete networks, the condition could mean one more scenario in which everyone weighs the maximal and minimal opinions in the same way.

Finally, we note here that our sufficient condition is satisfied when at least one of the following conditions are satisfied: $\sum_{k=1}^{\infty} \underline{\alpha}^{t_k,\theta} = \infty$ or $\sum_{k=1}^{\infty} \underline{\beta}^{t_k,\theta} = \infty$.

4 Discussion

We now consider how our condition translates to specific networks we considered in the previous section.

Example 1 (L_p-updating function)**.** *If every agent has the same updating function in (1), then the weights do not depend on the current opinion. Thus, let* $\underline{w}_i^t \equiv \min_{j \in C(i)} w_{ij}^t$, $\underline{w}^t \equiv \min_{i \in A} \underline{w}_i^t$, *and* $\underline{w}^{t,\theta} \equiv \prod_{\tau=t}^{t+\theta-1} \underline{w}^\tau$ *for all* $t \geq 0$. *In this case, our sufficient condition is satisfied if there exists* $\{t_k\}$ *with* $t_{k+1} - t_k \geq \theta$ *and* $\sum_k \underline{w}^{t_k,\theta} = \infty$. *To see this, observe that when* $p = 1$ *we have that* $\underline{\alpha}^t = \underline{\beta}^t = \underline{w}^t$. *Thus,* $\sum_k \underline{w}^{t_k,\theta} = \infty$ *is equivalent to* $\sum_k (\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}) = \infty$. *Let* $p \in (0,1)$. *Then we know that by Jensen's inequality,*

$$\sum_{j \in C(i)} w_{ij}^t x_j^p \leq \left(\sum_{j \in C(i)} w_{ij}^t x_j \right)^p .$$

Subsequently,

$$T_i(t,x) = \left(\sum_{j\in C(i)} w_{ij}^t x_j^p\right)^{1/p} \leq \sum_{j\in C(i)} w_{ij}^t x_j \leq \underline{w}^t \min_{j\in C(i)} x_j + (1-\underline{w}^t) \max_{j\in C(i)} x_j.$$

Thus, $\sum_k \underline{w}^{t_k,\theta} = \infty$ *implies that* $\sum_k \underline{\beta}^{t_k,\theta} = \infty$. *A similar proof works for the* $p > 1$ *or* $p < 0$ *cases.*

Example 2. *If every agent's updating function is the form of (3), then the weights do not depend on the current opinion. In this case, we will show that, unless the initial opinions satisfy both* $\min_{i\in A} x_i^0 = 0$ *and* $\max_{i\in A} x_i^0 = 1$, *consensus is reached. Clearly, if either* $\min_{i\in A} x_i^0 = 0$ *or* $\max_{i\in A} x_i^0 = 1$ *(but not both) then opinions converge to 0 or 1, respectively (for irreducible, aperiodic networks). Thus, let us concentrate on opinions where* $0 < \min_{i\in A} x_i < \max_{i\in A} x_i \in (0,1) < 1$.

As in the previous example let us define $\underline{w}^t$ *and* $\underline{w}^{t,\theta}$ *for all* $t \geq 0$. *In this case, our sufficient condition is satisfied if there exists* $\{t_k\}$ *with* $t_{k+1} - t_k \geq \theta$ *and* $\sum_k \underline{w}^{t_k,\theta} = \infty$.

To prove this, let $Z_i^t = \frac{x_i^t}{(1-x_i^t)}$ *and* $z_i^t = lnZ_i^t$. *Notice that the updating function can be rewritten as* $Z_i^{t+1} = \Pi_{j\in C(i)}(Z_j^t)^{w_{ij}^t}$ *and therefore:*

$$z_i^{t+1} = \sum_{j\in C(i)} w_{ij}^t z_i^t$$

Notice that this has the same structure as time-varying DeGroot (1974), which is a special case of (1). The only difference is that $z_i^t \in (-\infty,+\infty)$, *which is not a compact set. However, our proof for Theorem 1 is valid when* $[\min_{i\in A} z_i^0, \max_{i\in A} z_i^0]$ *is a compact set, which occurs when* $\min_{i\in A} x_i^0, \max_{i\in A} x_i^0 \in (0,1)$. *Thus, consensus is reached as long as there exists* $\{t_k\}$ *with* $t_{k+1} - t_k \geq \theta$ *and* $\sum_k \underline{w}^{t_k,\theta} = \infty$ *as we have shown in Example 1.*

Finally, let us consider the updating functions in the form of (2). DeMarzo et al. (2003) consider show that consensus is reached if $\sum_{t=1}^{+\infty} \lambda^t = +\infty$ in this setting. Our condition would require the existence of a sequence $\{t_k\}$ with $t_{k+1} - t_k \geq \theta$ and $\sum_{k=1}^{+\infty} \lambda^{t_k,\theta} = \infty$ where $\lambda^{t,\theta} \equiv \prod_{\tau=t}^{t+\theta-1} \lambda^\tau$. Thus, our condition is more restrictive than that of DeMarzo et al. (2003). The two conditions however are equivalent in complete networks. This observation raises the following question: can our condition be replaced in Theorem 1 by $\sum_{t=1}^{+\infty}(\underline{\alpha}^t + \underline{\beta}^t) = \infty$. The answer turns out to be negative and we demonstrate this point in the example below.

Example 3. *There are four agents and agent 1 listens to agents 1 and 2, agent 2 to agents 1, 2 and 3, agent 3 to agents 2, 3 and 4, and agent 4 to agents 3 and 4. The updating* *functions are as follows (for $\epsilon < \frac{1}{4}$):*

$$T_i(t,x) = \begin{cases} (1-\delta_i^t)\min_{j\in C(i)} x_j + \delta_i^t \max_{j\in C(i)} x_j & \textit{if } i = 1,2 \\ \delta_i^t \min_{j\in C(i)} x_j + (1-\delta_i^t)\max_{j\in C(i)} x_j & \textit{if } i = 3,4 \end{cases}$$

where

$$\delta_1^t = \delta_4^t = \begin{cases} \frac{\epsilon}{2(2^{t-2}-(2^t-1)\epsilon)} & \textit{if } t \textit{ is even} \\ 0.5 & \textit{if } t \textit{ is odd} \end{cases}$$

and

$$\delta_2^t = \delta_3^t = \begin{cases} \frac{\epsilon}{2^{t-2}-(2^t-1)\epsilon} & \textit{if } t \textit{ is even.} \\ \frac{2^{t-2}-\left(2^t-1\right)\epsilon}{2^{t-1}-(2^{t+1}-1)\epsilon} & \textit{if } t \textit{ is odd} \end{cases}$$

Let us consider the sequence $\{T^{0,t}(0,0.5,0.5,1)\}$. One can calculate that

$$T^{0,t}(0,0.5,0.5,1) = \begin{cases} \left(\frac{2^t-1}{2^{t-1}}\epsilon, 2\epsilon, 1-2\epsilon, 1-\frac{2^t-1}{2^{t-1}}\epsilon\right) & \textit{if } t \textit{ is odd} \\ \left(\frac{2^t-1}{2^{t-1}}\epsilon, \frac{1}{2}, \frac{1}{2}, 1-\frac{2^t-1}{2^{t-1}}\epsilon\right) & \textit{if } t \textit{ is even.} \end{cases}$$

One can easily see that the first and last agent's opinion converges to 2ϵ and $1-2\epsilon$, respectively. However, the opinions of agents 2 and 3 do not converge.

Observe here that $\underline{\alpha}^t = \underline{\beta}^t = \min_{i=1,\cdots,4}\{\delta_i^t\}$. Furthermore, $\sum_t \underline{\alpha}^t = \sum_t \underline{\beta}^t = \infty$ because the even numbered $\underline{\alpha}^t$s and $\underline{\beta}^t$s converge to 0 while the odd numbered ones to 0.5. However, as we already mentioned above, the agents do not converge to a consensus. Our sufficient condition is not satisfied here. To see this, observe that $\theta = 3$ in this example. Thus, any three consecutive periods will have at least one odd period and $\underline{\alpha}^t$ and $\underline{\beta}^t$ decrease by 4th between any two consecutive odd periods. Subsequently, whatever 3 period blocks we choose, both $\underline{\alpha}^{t,\theta}$ and $\underline{\beta}^{t,\theta}$ decrease at least by half between two blocks, which is a too fast of a decrease.

The above example has a very specific structure: Let us focus on agents 1 and 2 because their behavior is copied by the other two in the opposite way. Agent 1 has the minimal opinion in all periods which increases over time. Agent 2's opinions bounce between 0.5 and some values which get increasingly closer to agent 1's opinion. This alternating feature of agent 2's opinion is justified because agent 3's opinions also bounce around and counterbalance. The reason for non-convergence of opinions is the following: agent 1 increasingly relies on her own opinion over time in odd periods, i.e., whenever the opinions of agents 1 and 2 are farther apart. Hence, agent 1's opinion moves very little from where it was in these periods. On the other hand, agent 1 gives almost the equal weights to her and 2's opinions in even periods, i.e., whenever the opinions of agents 1 and 2 are very close. Unfortunately, in these periods their opinions are increasingly closer; thus, agent 1's opinion barely budges. Consequently, the opinions do not converge.

We conclude this note by considering the restriction of our analysis to Markov chains. Are our results robust to conditioning on the full history of beliefs? Notice that Theorem 1 is valid for history-dependent updating functions. In fact, the proof remains the same. Updating functions that violate the Markov property converge to consensus if the conditions of the theorem are satisfied.

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Appendix

To prove Lemma 1 we first introduce some notation and definitions. Let $\underline{T} : [0,1]^n \to [0,1]^n$ $\underline{T} : [0,1]^n \to [0,1]^n$ be a function such that

$$\underline{T}_i(t,x) = (1 - \underline{\alpha}^t) \min_{j \in C(i)} x_j + \underline{\alpha}^t \max_{j \in C(i)} x_j$$

for all i and x. We define $\underline{T}^{t+\tau,t}(x)$ in the same way as we defined $T^{t+\tau,t}(x)$.

The following lemma plays a key role in the proof of Lemma 1.

Lemma 2. *(a) For all natural number $\tau \geq 1$ and $t \geq 0$, $\underline{T}^{t+\tau,t}(x)$ is monotonic.*

(b) For all natural number $k \geq 1$ and x, $T^{t,k}(x) \geq \underline{T}^{t,k}(x)$.

(c) Let $\langle A, C\rangle$ be irreducible and aperiodic. Then for any x and $j \in A$,

$$\min_{j\in A} T_j^{t,\theta}(x) \geq (1-\underline{\alpha}^{t,\theta}) \min_{j\in A} x_j + \underline{\alpha}^{t,\theta} \max_{j\in A} x_j.$$

Proof. (a) Because

$$\underline{T}_i(t,x) = \underline{\alpha}^t \max_{j\in C(i)} x_j + (1-\underline{\alpha}^t) \min_{j\in C(i)} x_j,$$

we have $\underline{T}(x) \geq \underline{T}(x^*)$ whenever $x \geq x^*$. Furthermore, the monotonicity of $\underline{T}(\tau, x)$ for all τ and the definition of $\underline{T}^{t+\tau,t}(\cdot)$ imply that $\underline{T}^{t+\tau,t}(\cdot)$ is monotonic.
(b) By the definition of $\underline{T}(t,x)$, we have that $T(\tau, y) \geq \underline{T}(\tau, y)$ for all non-negative natural number τ and y. Subsequently, $T(t,x) \geq \underline{T}(t,x)$ and $T(t+1, T(t,x)) \geq \underline{T}(t+1, T(t,x))$ for all t. By combining these with the monotonicity of $\underline{T}(t,\cdot)$, we obtain that

$$T^{t,2}(x) = T(t+1, T(t,x)) \geq \underline{T}(t+1, T(t,x)) \geq \underline{T}(t+1, \underline{T}(t,x)) = \underline{T}^{t,2}(x).$$

One can extend the argument above and obtain that

$$T^{t,k}(x) \geq \underline{T}^{t,k}(x)$$

for each natural number $k \geq 1$.
(c) Recall that θ satisfies the following condition: $j \in C^\theta(i)$ for all $i, j \in A$. We now show that for any x,

$$\min_{j\in A} \underline{T}_j^{t,\theta}(x) \geq (1-\underline{\alpha}^{t,\theta}) \min_{j\in A} x_j + \underline{\alpha}^{t,\theta} \max_j x_j.$$

Let $\bar{i}$ be an agent for whom $x_{\bar{i}} = \max_{j\in A} x_j$. We know that each $i \in A$ listens to $\bar{i}$ in θ steps, i.e., $\bar{i} \in C^\theta(i)$. Let y be an opinion such that $y_i = \min_{j\in A} x_j$ for all $i \neq \bar{i}$ and $y_{\bar{i}} = x_{\bar{i}} = \max_{j\in A} x_j$. Clearly, $x \geq y$. Thus, by the monotonicity of $\underline{T}^{t,\tau}(\cdot)$,

$$\underline{T}^{t,\tau}(x) \geq \underline{T}^{t,\tau}(y)$$

for all τ. We now concentrate on $\underline{T}(t,y)$. If i does not listen to $\bar{i}$ (i.e., if $\bar{i} \notin C(i)$), then $\underline{T}_i(t,y) = \min_{j\in A} x_j$. On the other hand, if i listens to only $\bar{i}$ (i.e., $\{\bar{i}\} = C(i)$), then $\underline{T}_i(t,y) = y_{\bar{i}} = \max_{j\in A} x_j$. If i listens to some other agents in addition to $\bar{i}$ (i.e., $\{\bar{i}\} \subset C(i)$), then

$$\begin{aligned}\underline{T}_i(t,y) &= (1-\underline{\alpha}^t) \min_{j\in C(i)} y_j + \underline{\alpha}^t \max_{j\in C(i)} y_j \\ &= (1-\underline{\alpha}^t) \min_{j\in A} x_j + \underline{\alpha}^t \max_{j\in A} x_j.\end{aligned}$$

Let y^1 be an opinion such that $y_i^1 = \min_{j\in A} x_j$ if $\bar{i} \notin C(i)$ and $y_i^1 = (1-\underline{\alpha}^t)\min_{j\in A} x_j + \underline{\alpha}^t \max_{j\in A} x_j$ if $\bar{i} \in C(i)$. Observe that $\underline{T}(t,y) \geq y^1$ for all i. Thus, by the monotonicity of $\underline{T}^{t,\tau}(\cdot)$ for all τ, $\underline{T}^{t,2}(x) \geq \underline{T}^{t,2}(y) \geq \underline{T}(t+1,y^1)$. We now turn our attention to $T(t+1,y^1)$. If i does not listen to $\bar{i}$ in two steps (i.e., if $\bar{i} \notin C^2(i)$), then $\underline{T}_i(t+1,y^1) = \min_{j\in A} x_j$. On the other hand, if i listens to $\bar{i}$ in two steps (i.e., $\bar{i} \in C^2(i)$), then

$$
\begin{aligned}
\underline{T}_i(t+1,y^1) &= (1-\underline{\alpha}^{t+1})\min_{j\in C(i)} y_j^1 + \underline{\alpha}^{t+1}\max_{j\in C(i)} y_j^1 = \\
&\geq (1-\underline{\alpha}^{t+1})\min_{j\in A} x_j + \underline{\alpha}^{t+1}((1-\underline{\alpha}^t)\min_{j\in A} x_j + \underline{\alpha}^t \max_{j\in A} x_j) \\
&= (1-\underline{\alpha}^{t,2})\min_{j\in A} x_j + \underline{\alpha}^{t,2}\max_{j\in A} x_j
\end{aligned}
$$

Let y^2 be an opinion such that $y_i^2 = \min_{j\in A} x_j$ if $\bar{i} \notin C^2(i)$ and $y_i^2 = (1-\underline{\alpha}^{t,2})\min_{j\in A} x_j + \underline{\alpha}^{t,2}\max_{j\in A} x_j$ if $\bar{i} \in C^2(i)$. Observe that $\underline{T}(t+1,y^1) \geq y^2$. Thus, by the monotonicity of $\underline{T}^{t,\tau}(\cdot)$ for all τ, $\underline{T}^{t,3}(x) \geq \underline{T}^{t,3}(y) \geq \underline{T}^{t+1,2}(y^1) \geq \underline{T}(t+2,y^2)$. We now turn our attention to $\underline{T}(t+2,y^2)$. If i does not listen to $\bar{i}$ in three steps (i.e., if $\bar{i} \notin C^3(i)$), then $\underline{T}_i(t+2,y^2) = \min_{j\in A} x_j$. On the other hand, if i listens to $\bar{i}$ in three steps (i.e., $\bar{i} \in C^3(i)$), then

$$
\begin{aligned}
\underline{T}_i(t+2,y^2) &= \min_{j\in C(i)} y_j^2 + \underline{\alpha}^{t+2}\left(\max_{j\in C(i)} y_j^2 - \min_{j\in C(i)} y_j^2\right) \\
&\geq (1-\underline{\alpha}^{t,3})\min_{j\in A} x_j + \underline{\alpha}^{t,3}\max_{j\in A} x_j.
\end{aligned}
$$

By following the same procedure iteratively, let us define $y_i^{\theta-1}$. Observe that $\underline{T}(t+\theta-2, y^{\theta-2}) \geq y^{\theta-1}$. Thus, by the monotonicity of $\underline{T}^{t,\tau}$ for all τ, $\underline{T}^{t,\theta}(x) \geq \underline{T}^{t,\theta}(y) \geq \underline{T}^{t+1,\theta-1}(y^1) \geq \cdots \geq \underline{T}^{t+\theta-1,1}(y^{\theta-1}) = \underline{T}(\theta-1,y^{\theta-1})$. We now turn our attention to $T(\theta-1,y^{\theta-1})$. We know that each i listens to $\bar{i}$ in θ periods. Thus,

$$
\begin{aligned}
\underline{T}_i(\theta-1,y^{\theta-1}) &= \min_{j\in C(i)} y_j^{\theta-1} + \underline{\alpha}^{\theta-1}\left(\max_{j\in C(i)} y_j^{\theta-1} - \min_{j\in C(i)} y_j^{\theta-1}\right) \\
&\geq (1-\underline{\alpha}^{t,\theta})\min_{j\in A} x_j + \underline{\alpha}^{t,\theta}\max_{j\in A} x_j
\end{aligned}
$$

This means that $\min_{i\in A} \underline{T}_i^{t,\theta}(x) \geq (1-\underline{\alpha}^{t,\theta})\min_{j\in A} x_j + \underline{\alpha}^{t,\theta}\max_{j\in A} x_j$. By combining this with (b) of this lemma, we obtain (c). □

Proof of Lemma 1. Parts b and c of Lemma 2 yield that

$$\min_{i\in A} T_i^{t,\theta}(x) \geq (1-\underline{\alpha}^{t,\theta}) \min_{j\in A} x_j + \underline{\alpha}^{t,\theta} \max_{j\in A} x_j.$$

Similarly, one can show that

$$\max_{i\in A} T_i^{t,\theta}(x) \leq \underline{\beta}^{t,\theta} \min_{j\in A} x_j + (1-\underline{\beta}^{t,\theta}) \max_{j\in A} x_j.$$

Consequently,

$$\max_{i\in A} T_i^{t,\theta}(x) - \min_{i\in A} T_i^{t,\theta}(x) \leq (1-\underline{\alpha}^{t,\theta}-\underline{\beta}^{t,\theta})(\max_{j\in A} x_j - \min_{j\in A} x_j).$$

□

Proof of Theorem 1. Fix any x. Set $x^0 = x$ and $x^t = T^{0,t}(x)$ for all $t \geq 1$. Now consider the sequence $\{x^t\}$. Let $\underline{x}^t = \min_{i\in A} x_i^t$ and $\bar{x}^t = \max_{i\in A} x_i^t$. To prove the theorem it suffices to show $\lim_{t\to\infty}\{\bar{x}^t - \underline{x}^t\} \to 0$. Because $T_i(\tau, x) \in [\min_{j\in A} x_j, \max_{j\in A} x_j]$ for all i and τ, $\{\bar{x}^t - \underline{x}^t\}$ is a non-increasing sequence. Thus, we only need to show that the distance between extremal opinions converges to 0 for some subsequence. Let $\{t_k\}$ be a subsequence with $t_{k+1} - t_k \geq \theta$ for all k and $\lim_{\tau\to\infty} \sum_{k=1}^{\tau}(\underline{\alpha}^{t_k} + \underline{\beta}^{t_k}) = \infty$. Because $\{\bar{x}^t - \underline{x}^t\}$ is non-increasing and $\{t_k\}$ satisfies $t_{k+1} - t_k \geq \theta$ for all k, Lemma 1 gives that for all $\tau \geq 2$,

$$\begin{aligned}
\bar{x}^{t_\tau} - \underline{x}^{t_\tau} &\leq \max_{i\in A} T_i^{t_{\tau-1},\theta}(x^{t_\tau-1}) - \min_{i\in A} T_i^{t_{\tau-1},\theta}(x^{t_{\tau-1}}). \\
&\leq (1-\underline{\alpha}^{t_{\tau-1},\theta} - \underline{\beta}^{t_{\tau-1},\theta})\,(\bar{x}^{\tau_{\tau-1}} - \underline{x}^{\tau_{\tau-1}}.) \\
&\leq \prod_{k=1}^{\tau-1}(1-\underline{\alpha}^{t_k,\theta} - \underline{\beta}^{t_k,\theta})\left(\bar{x}^0 - \underline{x}^0\right).
\end{aligned}$$

Consequently, we complete the proof by showing that $\lim_{\tau\to\infty}\prod_{k=1}^{\tau}(1-\underline{\alpha}^{t_k,\theta} - \underline{\beta}^{t_k,\theta}) \to 0$ when $\lim_{\tau\to\infty}\sum_{k=1}^{\tau}(\underline{\alpha}^{t_k} + \underline{\beta}^{t_k}) = \infty$. It is easy to see that for any $\tau \geq 1$,

$$\prod_{k=1}^{\tau}(1-\underline{\alpha}^{t_k,\theta} - \underline{\beta}^{t_k,\theta}) \leq \left(1 - \frac{\sum_{k=1}^{\tau}\left(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}\right)}{\tau}\right)^{\tau}.$$

In addition,

$$\lim_{\tau\to\infty}\left(1 - \frac{\sum_{k=1}^{\tau}\left(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}\right)}{\tau}\right)^{\tau} \leq \exp\left(-\sum_{k=1}^{l}\left(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}\right)\right) \;\; \forall l \in \mathbb{N}.$$

□

Furthermore, because $\lim_{l\to\infty}\sum_{k=1}^{l}\left(\underline{\alpha}^{t_k,\theta}+\underline{\beta}^{t_k,\theta}\right)=\infty$, the previous three inequalities give that

$$\lim_{\tau\to\infty}\prod_{k=1}^{\tau}(1-\underline{\alpha}^{t_k,\theta}-\underline{\beta}^{t_k,\theta})=0.$$